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Chaos for the Sierpinski Carpet

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We study the chaotic behavior of the Sierpinski carpet. It is proved that this dynamical system has a chaotic set whose Hausdorff dimension equals that of the Sierpinski carpet.

KEY WORDS: Fractal; chaos; Hausdorff dimension; thermodynamic formalism; strong-mixing; outer measure.

1. INTRODUCTION

Recently, chaotic dynamical systems on fractals have been studied by M. Barnsley and D. Simpelaere. Simpelaere[9] studied the recurrence and return times of orbits of points of a dynamical system, the *Sierpinski carpet*, and proved that the Poisson law property holds almost everywhere with respect to a natural measure defined on this system. Here we study the chaotic behavior of this dynamical system, and calculate the Hausdorff dimension of a chaotic set of the Sierpinski carpet.

Li and Yorke[5] originally introduced the notion of chaos for continuous self-maps of the interval $I=[0,1]$, and showed that if a continuous map $f\colon I\mapsto I$ has a periodic point with period three, then the following condition (*) is satisfied:

(*) There exists an uncountable subset C of I such that for any different points y_1 and y_2 of C,

$$\liminf_{i\to\infty}|f^i(y_1)-f^i(y_2)|=0 \qquad \text{and} \qquad \limsup_{i\to\infty}|f^i(y_1)-f^i(y_2)|>0$$

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i.e., there exist two increasing sequences $\{m_i\}$ and $\{k_i\}$ of positive integers such that

$$\lim_{i\to\infty} f^{m_i}(y_1) \neq \lim_{i\to\infty} f^{m_i}(y_2) \qquad \text{and} \qquad \lim_{i\to\infty} f^{k_i}(y_1) = \lim_{i\to\infty} f^{k_i}(y_2)$$

The set C is called a chaotic set of f in the sense of Li and Yorke.

In the above theorem, Li and Yorke used the cardial to describe the size of the chaotic set C of f. The notion of Hausdorff dimension plays an important role in describing the concept of size of sets. The Hausdorff dimension is an active research subject in contemporary ergodic theory. The main aim of the present paper is to show that a dynamical system, the Sierpinski carpet, has a chaoticity suggested in ref. 13 which is more complicated than the chaoticity in the sense of Li and Yorke and that the Sierpinski carpet has a chaotic set whose Hausdorff dimension equals the Hausdorff dimension of the Sierpinski carpet.

Definition 1. Let X be a topological space and $f: X \mapsto X$ be a map. Let $\{p_i\}$ he any strictly increasing sequence of positive integers. $S \subset X$ is called a chaotic set of f with respect to $\{p_i\}$ if, for any finite subset $A \subset S$ and any map $F: A \mapsto X$, there is a subsequence $\{r_i\}$ of the sequence $\{p_i\}$ such that

$$\lim_{i\to\infty} f^{r_i}(x) = F(x)$$

for all $x \in A$.

Remark 1. If $S \subset X$ is a chaotic set of f with respect to $\{p_i\}$, then S is a chaotic set of f in the sense of Li and Yorke.

The following theorem has been proved in ref. 15.

Theorem A. Let X be a topological space satisfying the second axiom of countability, $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X, and let μ be a probability measure on $(X, \mathscr{B}(X))$ with the property that each nonempty open set has nonzero measure. Suppose $f: X \mapsto X$ is a transformation which preserves the measure μ and is strong-mixing. Then for any strictly increasing sequence $\{p_i\}$ of positive integers, there is a chaotic set $C \subset X$ with respect to $\{p_i\}$ such that for any $D \in \mathscr{B}(X)$, if $\mu(D) > 0$, then $C \cap D \neq \varnothing$.

2. THE CHAOTIC MODEL

Given integers $m \geqslant n$ and a set

$$S \subset \{(i, j): 0 \leqslant i < n \text{ and } 0 \leqslant j < m\}$$

with $\#(S) = r > 1$, define the fractal set $\bar{S}$ by

$$\bar{S} = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right): (x_k, y_k) \in S, \forall k \geqslant 1 \right\}$$

It is clear that $\bar{S} = \bigcup_{k=0}^{r-1} f_k(\bar{S})$, where the f_k are affine maps contracting by a factor of n horizontally and m vertically, i.e., for k which corresponds to a pair $(i, j) \in S$,

$$f_k(x, y) = \left(\frac{i+x}{n}, \frac{j+y}{m} \right) \qquad \text{for} \quad (x, y) \in [0, 1]^2$$

Barnsley[1] called $\{f_1, f_2, ..., f_r\}$ an iterated function system (IFS); $\bar{S}$ is an attractor of the IFS.

We also have a continuous surjective map ϕ: $S_r = \{0, 1, ..., r-1\}^N \mapsto \bar{S}$:

$$(i_1, i_2, ...) \mapsto \bigcap_{j \geqslant 1} f_{i_1} f_{i_2} \cdots f_{i_j}([0, 1]^2)$$

Then there exists a set $M \subset S_r$ such that:

(1) $\sigma(M) \subset M$, where σ: $S_r \mapsto S_r$ is the shift map defined by

$$\sigma(\alpha_1, \alpha_2, ...) = (\alpha_2, \alpha_3, ...)$$

(2) $\phi|_M$ is one-to-one.

(3) $\rho(M) = 1$ for any product measure ρ defined by a probability vector $(p_0, p_1, ..., p_{r-1})$.

So we get a map f: $\phi(M) \mapsto \phi(M)$ by

$$f(x) = f_i^{-1}(x) \qquad \text{if} \quad x \in \phi(M) \cap f_i(\bar{S})$$

It is easy to see that $f\phi = \phi f$.

Barnsley[1] also defined a dynamical system on fractals as follows:

A sequence of points $\{x_n\}_{n=0}^{\infty}$ in $\bar{S}$ is called an orbit of the random shift dynamical system associated to the Sierpinski carpet if

$$x_{n+1} = \begin{cases} f_i^{-1}(x_n) & \text{when } x_n \in f_i(\bar{S}) \setminus \bigcup_{j \neq i} f_j(\bar{S}) \\ \text{one of } \{f_{i_j}^{-1}(x_n)\}_{j=1}^{l} & \text{when } x_n \in \bigcap_{j=1}^{l} f_{i_j}(\bar{S}) \end{cases}$$

for each $n \in \{0, 1, 2, ...\}$.

Remark 2. $\{x_n\}_{n=0}^{\infty}$ is an orbit of the random shift dynamical systems associated to the Sierpinski carpet if and only if there exists a point $\alpha = (i_1, i_2, ...) \in S_r$ such that $x_n = \phi(\sigma^n(\alpha))$ for each $n \in \{0, 1, 2, ...\}$.

Remark 3. If $x_0 \in \phi(M)$, then $\{x_n\}_{n=0}^{\infty}$ is an orbit of the random shift dynamical systems associated to the Sierpinski carpet if and only if $x_{n+1} = f(x_n)$ for each $n \in \{0, 1, 2, ...\}$.

Now we state and prove the main result of the present paper.

Theorem 1. Let $\{p_i\}$ be any strictly increasing sequence of positive integers, then there is a subset W of $\bar{S}$ such that:

(i) For any finite subset $A \subset W$ and any map $F: A \mapsto \bar{S}$ there is a subsequence $\{r_i\}$ of the sequence $\{p_i\}$ such that

$$\lim_{i \to \infty} f^{r_i}(x) = F(x)$$

for all $x \in A$.

(ii) $\dim_H(W) = \dim_H(\bar{S}) = \log_m(\sum_{j=0}^{m-1} t_j^{(\log_n m)})$, where t_j is the number of i such that $(i, j) \in S$.

Proof. Following ref. 7, let $\delta = \log_m(\sum_{j=1}^{m-1} t_j^{(\log_n m)})$ and let $(x_i, y_i)_{i=0}^{r-1}$ enumerate the elements of S. For $i = 0, 1, ..., r-1$, let a_i be the number of j such that $y_i = y_j$; then

$$m^{\delta} = \sum_{0}^{r-1} a_i^{(\log_n m - 1)}$$

Put $b_i = a_i^{(\log_n m - 1)}/m^{\delta}$, then $\sum_0^{r-1} b_i = 1$. By ref. 9, the one-side $(b_0, b_1, ..., b_{r-1})$-shift is strong-mixing, it follows from Theorem A that there is a chaotic set of σ such that for any Borel subset D with $\mu(D) > 0$, then $C \cap D \neq \varnothing$. Let μ^* be the outer measure induced from μ.

Claim 1. $\mu^*(C) = 1$.

It is easy to see that μ^* is a regular outer measure. If $\mu^*(C) < 1$, then there exists a set $F \in \mathscr{B}(S_r)$ such that $C \subset F$ and $\mu^*(C) = \mu(F) < 1$; then $\mu(S_r \backslash F) > 0$ and $(S_r \backslash F) \cap C \neq \varnothing$. This contradicts Theorem A.

Define the function g_k on S_r,

$$g_k(i_1 i_2 \cdots) = \left[\frac{(a_{i_1} a_{i_2} \cdots a_{i_k})^{\log_n m}}{(a_{i_1} a_{i_2} \cdots a_{i_l})} \right]^{1/k}$$

where $l = [k \log_n m]$ as usual. By Lemma 4 of ref. 6, we have that

$$\mu\{z \in S_r \colon g_k(z) \to 1\} = 1$$

Let $E = C \cap M \cap \{z \in S_r \colon g_k(z) \to 1\}$; then $\mu^*(E) = 1$. Let $W = \phi(E)$; the W is a chaotic set of $\{\bar{S}, f\}$.

Claim 2. $\dim_H(W) = \delta$.

Fix $\beta < \delta$; let

$$E_K = \{z \in E \colon g_k(z) < m^{\delta - \beta} \text{ for all } k \geqslant K\}$$

Since μ^* is a regular outer measure, we have that

$$\mu(E_K) \to \mu^*(E) = 1 \qquad (K \to \infty)$$

So we can pick K such that $\mu^*(E_K) > 0$. Set $\varepsilon = \min\{\mu^*(E_K), m^{-\beta K}\}$.

To a covering $C = \{A_k(p, q) \prod B_k\}$ defined in ref. 7, let N_k be the number of $A_{k'}(p, q) \prod B_{k'} \in C$ with $k' = k$. If $N_k \neq 0$ for some $k > K$, then $\sum N_k m^{-\beta k} > m^{-\beta k} > \varepsilon$. So assume that $N_k = 0$ for $k < K$; then for the elements of C such that $(A_k(p, q) \prod B_k) \cap E_k \neq \varnothing$, we have

$$\mu\left(A_k(p, q) \prod B_k\right) = [g_k(z) m^{-\delta}]^k < m^{-\beta k}$$

where $z \in (A_k(p, q) \prod B_k) \cap E_k$ and $k > K$. Since C covers E_k, we have

$$\sum N_k m^{-\beta k} > \mu(E_k) \geqslant \varepsilon$$

By Lemma 2 of ref. 7, $\mu_\beta(W) > 0$; it follows that $\dim_H(W) \geqslant \delta$. On the other hand, by the main theorem of ref. 7, we have that $\dim_H(W) \leqslant \dim_H(\bar{S}) = \delta$. The proof is complete.

Corollary 1. For the Sierpinski carpet, there is a subset W of $\bar{S}$ such that:

(i) For $x, y \in W, x \neq y$, we have

$$\liminf_{i \to \infty} |f^i(x) - f^i(y)| = 0 \qquad \text{and} \qquad \limsup_{i \to \infty} |f^i(x) - f^i(y)| = |\bar{S}|$$

(ii) $\dim_H(W) = \dim_H(\bar{S}) = \log_m(\sum_{j=0}^{m-1} t_j^{(\log_n m)})$, where t_j is the number of i such that $(i, j) \in S$.

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